Quasi maximum likelihood blind deconvolution: super- and sub-Gaussianity vs. asymptotic stability

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***Abstract*— In this note we consider the problem of quasi maximum likelihood (QML) blind deconvolution. We examine two classes of estimators, which are commonly believed to be suitable for super- and sub-Gaussian sources. We state the asymptotic stability conditions and demonstrate a distribution, for which the studied estimators result unsuitable, in the sense that they are asymptotically unstable.**

***Index Terms*— blind deconvolution, quasi maximum likelihood, asymptotic stability, super-Gaussian, sub-Gaussian, kurtosis.**

I. Introduction

We consider the problem of blind deconvolution, in which the observed sensor signal x is created from the *source signal* s passing through a convolutive system with impulse response w,

$$x_n = \sum_{k=-\infty}^{\infty} w_k \; s_{n-k}.$$

The setup is termed *blind* when only x is accessible, whereas no knowledge on w and s is available. The problem of blind deconvolution aims to find such a deconvolution (or restoration) kernel h, that produces a possibly delayed waveform-preserving estimate of s:

$$\tilde{s}_n = \sum_{k=0}^{\infty} h_k \; x_{n-k} \approx c \cdot s_{n-\Delta},$$

where c is a scaling factor and Δ is an integer shift. A commonly used assumption is that s is non-Gaussian.

Under the assumption that the restoration kernel h has no zeros on the unit circle, and the source signal is real and i.i.d., the normalized log-likelihood function of the observed signal x in the noise-free case is [1]–[3]

$$\ell(x;h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|H(e^{i\theta})\right| d\theta - \frac{1}{T}\sum_{n=0}^{T-1}\varphi\left((x*h)_n\right), \quad (1)$$

where $H(e^{i\theta})$ stands for the discrete Fourier transform of h, and $\varphi(s) = -\log p(s)$, where $p(s)$ is the probability density function (PDF) of the source s_n. Consistent estimator can be obtained by maximizing $\ell(x;h)$ even when $\varphi(s)$ is not exactly equal to $-\log p(s)$. Such QML estimation has been shown to be practical in instantaneous blind source separation [4]–[7] and blind deconvolution [2], [8], [9] when the source PDF is unknown or not well-suited for optimization. Generally, $\ell(x;h)$ is maximized by gradient-based methods, hence, the main concern is the choice of $\varphi'(s)$.

It is commonly believed, that the knowledge of whether the source is super- or sub-Gaussian (i.e., such that its *kurtosis excess* defined by

$$\kappa = \frac{\mathbb{E}s^4}{\mathbb{E}^2 s^2} - 3$$

is either positive or negative, respectively) is sufficient in order to construct a consistent QML estimator. This belief leads to attributing $\varphi'(s)$ either to the class of functions suitable for estimation of super-Gaussian sources, and not suitable for estimation of the sub-Gaussian ones, or vice versa. For example, it is usually assumed (see e.g. [1], [2], [10], [11]) that the choice of the smoothed sign function, e.g.,

$$\varphi'(s) = \tanh(\beta s), \quad (2)$$

for $\beta \geq 1$, leads to a QML estimator suitable for super-Gaussian sources. Another example is the family of functions

$$\varphi'(s) = |s|^{\mu}\mathrm{sign}(s) \quad (3)$$

with the parameter $\mu > 1$, which is believed to be suitable for sub-Gaussian sources.

It this note, we state the conditions, under which a QML estimator is asymptotically stable, and show that generally there is no connection between the sign of kurtosis excess and asymptotic stability. We study the estimators obtained from (2), (3), for sources obeying the generalized Cauchy distribution. Although we focus our attention on the blind deconvolution problem, similar conclusions can be made regarding the blind source separation problem as well.

II. Asymptotic stability conditions

For a general choice of $\varphi'(s)$, the corresponding QML estimator is (locally) asymptotically stable if the following conditions hold [9]:

$$\mathbb{E}\varphi''(s) > 0 \quad (4)$$
$$\mathbb{E}^2\varphi''(s)\mathbb{E}^2(cs)^2 > 1 \quad (5)$$
$$\mathbb{E}\varphi''(cs)(cs)^2 + 1 > 0, \quad (6)$$

where c is a scaling factor, obtained from the solution of

$$\mathbb{E}\varphi'(cs)cs = 1. \quad (7)$$

These conditions are valid when the expected values $\mathbb{E}\varphi''(s)$, $\mathbb{E}\varphi''(s)s^2$, $\mathbb{E}\varphi'(s)$, and $\mathbb{E}s^2$ exist and are bounded. Similar stability conditions exist in the context of the blind source separation problem [12], [13].

When $\varphi'(s) = |s|^{\mu}\mathrm{sign}(s)$, it can be shown that

$$\begin{aligned} c &= \left((\mu+1)\cdot\mathbb{E}|s|^{\mu+1}\right)^{-1/(\mu+1)} \\ \mathbb{E}\varphi''(s)(cs)^2 &= \mu(\mu+1)c^{\mu+1}\cdot\mathbb{E}|s|^{\mu+1} \\ \mathbb{E}\varphi''(s) &= \mu(\mu+1)c^{\mu-1}\cdot\mathbb{E}|s|^{\mu-1}. \end{aligned}$$

For $\mu > 1$, it can be easily checked that conditions (4), (6) hold, hence, the asymptotic stability condition (5) yields

$$\Delta_s = \frac{\mathbb{E}|s|^{\mu+1}}{\mathbb{E}s^2 \; \mathbb{E}|s|^{\mu-1}} - \mu < 0. \quad (8)$$

In the particular case when $\mu = 3$, the latter condition becomes $\Delta_s = \kappa < 0$, meaning that the estimator is asymptotically stable for sub-Gaussian sources, and asymptotically unstable for the super-Gaussian ones.

When $\varphi'(s) = \tanh(\beta s)$, conditions (4), (6) hold for every $\beta \geq 1$, since $\varphi''(s) > 0$. Therefore, the asymptotic stability condition (5) rewrites

$$\Delta_s = 1 - \mathbb{E}\varphi''(s)\cdot\mathbb{E}(cs)^2 < 0. \quad (9)$$

In the case of a general β, derivation of analytic expression of Δ_s is complicated. However, in the limit $\beta \to \infty$, $\varphi'(s) \to \mathrm{sign}(s)$, and

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$\varphi''(s) \to 2\delta(s)$. Hence, for a large β,

$$\begin{aligned} \mathbb{E}\varphi''(s)(cs)^2 &= \mathbb{E}\varphi''(s)(cs)^2 \approx c^2\beta \int_{-1/\beta}^{+1/\beta} sp(s)\ ds \approx 0 \\ \mathbb{E}\varphi''(s) &= \mathbb{E}\varphi''(s) \approx \beta \int_{-1/\beta}^{+1/\beta} p(s)\ ds \approx 2p(0), \end{aligned}$$

where c is obtained by substituting $\varphi'(cs)cs \approx \mathrm{sign}(cs)cs$ into equation (7):

$$c \approx \frac{1}{\mathbb{E}|s|}.$$

Therefore, the estimator is asymptotically stable if

$$\Delta_s \approx \frac{\mathbb{E}|s|}{2p(0)\sigma^2} - 1 < 0. \tag{10}$$

In the limit $\beta \to \infty$, the latter condition is exact.

III. The generalized Cauchy distribution

Let us consider a parametric family of distributions with the parameters $a, r > 0$, described by the following PDF:

$$p(s) = \frac{ar^{1-\frac{1}{2a}} \sin\left(\frac{\pi}{2a}\right)}{\pi\left(|s|^{2a} + r\right)}$$

(see Figure 1). The parameter r influences the variance of s. For $a = 1$, one gets the Cauchy distribution; for this reason, this family of distributions will be henceforth referred to as the generalized Cauchy distribution.

It can be shown that the p-th moment of $|s|$ exists for $a > \frac{p+1}{2}$, and is given by

$$\mathbb{E}|s|^p = r^{\frac{p}{2a}} \cdot \csc\left(\frac{(p+1)\pi}{2a}\right) \sin\left(\frac{\pi}{2a}\right),$$

where

$$\csc x = \frac{1}{\sin x}$$

is the cosecant function. Particularly, the fourth order moment exists for $a > 2.5$ and the kurtosis excess is given by

$$\kappa(a) = \csc\left(\frac{\pi}{2a}\right) \csc\left(\frac{5\pi}{2a}\right) \sin^2\left(\frac{3\pi}{2a}\right) - 3.$$

$\kappa(a)$ is monotonically decreasing as a function of a and crosses zero for $a \approx 3.3567$ (see Figure 2, solid). This means that the source is super-Gaussian for $2.5 < a < 3.3567$, and sub-Gaussian for $a > 3.3567$.

For $\varphi'(s) = \tanh(\beta s)$, in the limit $\beta \to \infty$, the stability condition is given by

$$\begin{aligned} \Delta_s &= \frac{\mathbb{E}|s|}{2p(0)\sigma^2} - 1 \\ &= \frac{\pi}{2a} \csc\left(\frac{\pi}{2a}\right) \csc\left(\frac{\pi}{a}\right) \sin\left(\frac{3\pi}{2a}\right) - 1 < 0, \end{aligned}$$

and is valid for $a > 1.5$. Observe that

$$\frac{d\Delta_s}{da} = \frac{\pi}{4a^2} \left(2 + \cos\left(\frac{\pi}{a}\right)\right) \csc^2\left(\frac{\pi}{2a}\right) \sec^2\left(\frac{\pi}{2a}\right),$$

where

$$\sec x = \frac{1}{\cos x}$$

is the secant function. Since the derivative of Δ_s w.r.t. a is strictly positive, Δ_s is monotonically increasing with a. Δ_s crosses zero at $a \approx 2.3379$ (see Figure 2, dashed). This means that the corresponding

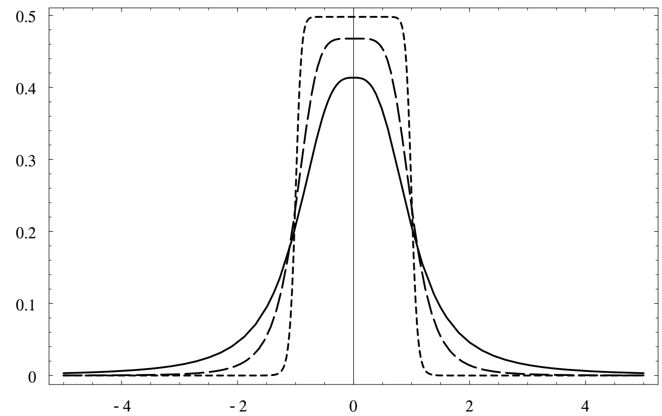

Fig. 1. PDF of the generalized Cauchy distribution for $r = 1$, $a = 1.5$ (solid), $a = 2.5$ (dashed), and $a = 10$ (dotted).

TABLE I
Asymptotic stability regions of different QML estimators

$\varphi'(s)$	**Asymptotic stability region**
$\tanh(s)$	$1.5 < a < 1.8666$
$\tanh(10s)$	$1.5 < a < 1.9344$
$\tanh(\beta s)$, $\beta \to \infty$	$1.5 < a < 2.3379$
$\|s\|^2\ \mathrm{sign}(s)$	$a > 3$
$\|s\|^3\ \mathrm{sign}(s)$	$a > 3.3567$
$\|s\|^4\ \mathrm{sign}(s)$	$a > 3.7352$

QML estimator is asymptotically unstable for $a > 2.3379$, particularly, the estimator is asymptotically unstable for both super- and sub-Gaussian sources. Δ_s was also evaluated numerically for $\beta = 1, 10$ (see Figure 3). Asymptotic stability regions of the estimators are presented in Table I.

For $\varphi'(s) = |s|^\mu \mathrm{sign}(s)$, the stability condition is given by

$$\begin{aligned} \Delta_s &= \frac{\mathbb{E}|s|^{\mu+1}}{\mathbb{E}s^2\ \mathbb{E}|s|^{\mu-1}} - \mu = \left(1 + 2\cos\left(\frac{\pi}{a}\right)\right) \cdot \\ &\quad \csc\left(\frac{\pi(\mu+2)}{2a}\right) \sin\left(\frac{\pi\mu}{2a}\right) - \mu < 0, \end{aligned}$$

and is valid for $a > 1 + \mu/2$. Observe that

$$\begin{aligned} \frac{d\Delta_s}{da} &= \frac{\pi}{2a^2} \csc\left(\frac{\pi(\mu+2)}{2a}\right) \left(2 + \cos\left(\frac{2\pi}{a}\right) \csc\left(\frac{2\pi}{a}\right) - \right. \\ &\quad \left. -\mu\left(1 + 2\cos\left(\frac{\pi}{a}\right)\right) \csc\left(\frac{\pi(\mu+2)}{2a}\right) \sin\left(\frac{\pi}{a}\right)\right) \end{aligned}$$

is negative for $a > 1 + \mu/2 > 1.5$ for every $\mu > 1$, and consequently, Δ_s is monotonically decreasing, with zero crossing depending on μ. Asymptotic stability regions for some values of μ are summarized in Table I, and the values of Δ_s are plotted as a function of a in Figure 2. Note that for $\mu = 3$, asymptotic stability is fully determined by the sign of kurtosis excess. However, this is not true for other values of μ.

IV. Conclusion

We have examined the asymptotic stability conditions for two classes of QML estimators, commonly used for super- and sub-Gaussian sources in blind deconvolution problems. The particular case of the generalized Cauchy distribution was examined. It can be concluded that asymptotic stability does not always correspond to the sign of kurtosis excess, which determines whether the source is super-

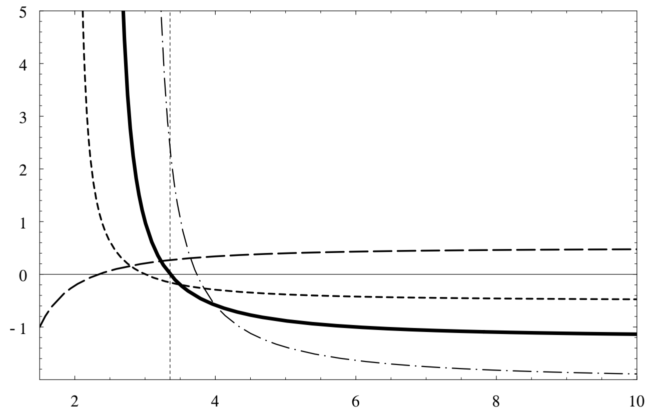

Fig. 2. The value of Δ_s as a function of a for different QML estimators: $\varphi'(s) = \mathrm{sign}(s)$ (dashed), $\varphi'(s) = |s|^2\mathrm{sign}(s)$ (dotted), $\varphi'(s) = |s|^3\mathrm{sign}(s)$ (solid), and $\varphi'(s) = |s|^4\mathrm{sign}(s)$ (dash-dotted). Kurtosis excess κ corresponds to Δ_s is the case $\varphi'(s) = |s|^3\mathrm{sign}(s)$. The estimator is stable for $\Delta_s < 0$.

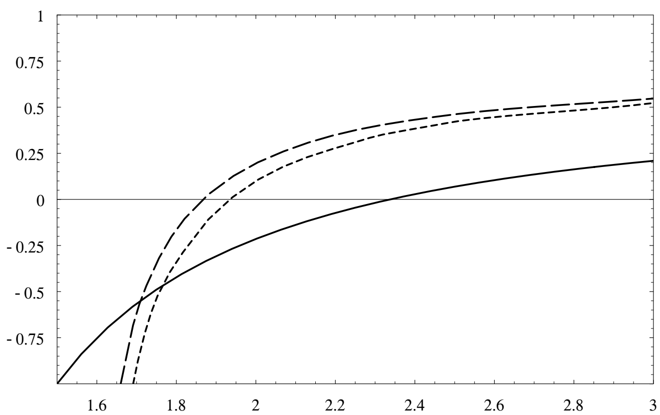

Fig. 3. The value of Δ_s as a function of a for the QML estimator $\varphi'(s) = \tanh(\beta s)$: $\beta = 1$ (dashed), $\beta = 10$ (dotted), and $\beta \to \infty$ (solid). The estimator is stable for $\Delta_s < 0$.

or sub-Gaussian. For example, the choice $\varphi'(s) = \tanh(\beta s)$, which is commonly believed to be suitable for super-Gaussian sources, is asymptotically unstable for such sources. The choice $\varphi'(s) = |s|^2\mathrm{sign}(s)$, which is known to be suitable for sub-Gaussian sources, is also suitable for some super-Gaussian sources (wherein $a > 3$). The choice $\varphi'(s) = |s|^4\mathrm{sign}(s)$, known to be suitable for sub-Gaussian sources, is asymptotically unstable for some of such sources (wherein $3.3567 < a < 3.7352$). With the only exception of $\varphi'(s) = |s|^3\mathrm{sign}(s)$, whose asymptotic stability is always determined by the sign of kurtosis excess, other QML estimators require more delicate analysis in order to determine whether they are suitable or not for estimation of super- or sub-Gaussian sources. Generally, the answer is distribution-dependent. The main conclusion from this note is that the non-linearity $\varphi'(s)$ should be chosen with more caution.

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